Manolescu

Today : advertisement

definition, example

Knot Floer homology

$K \subset S^3$ knot

simplest version

$\widehat{HFK}(K) = \bigoplus_{m,s \in \mathbb{Z}} \widehat{HFK}_m(K,s)$

bigraded abelian group

$\chi(\widehat{HFK})$

$= \sum (-1)^m g^s \operatorname{rank} \widehat{HFK}_m(K,s)$

$=$ Alexander polynomial $\Delta_K(K)$

m: Maslov (homological) grading

s: Alexander (filtration) grading

Properties

① $\text{genus}(K) \overset{\text{Ozsváth–Szabó}}{=} \max\{ s \mid \widehat{HFK}_*(K,s) \neq 0 \}$

$ \| $

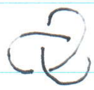
$\min\{ g \mid \Sigma \subset S^3,\ \partial\Sigma = K,\ \text{genus } \Sigma = g \}$

Hence $\widehat{HFK}$ detects the unknot

② K: fibered $\overset{\text{Ni}}{\iff} \operatorname{rk} \widehat{HFK}_*(K, \text{genus}(K)) = 1$

i.e. $F \to S^3 \setminus K$
$\downarrow$
S^1

Hence (Ghiggini)

$\widehat{HFK}$ detects  3 and 4_1

③ There are other versions HFK^-, HFK^+, CFK^∞ . . .

These encode the Heegaard Floer homologies of Dehn surgeries on K.

Original def. of HFK uses pseudoholomorphic curves
(Ozsvath - Szabo)
Rasmussen 2002

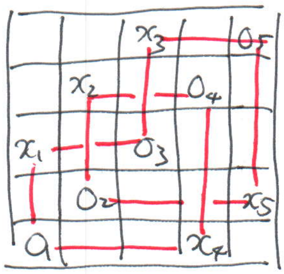
Combinatorial description M. Ozsvath, Sarkar (2006)
M. Ozsvath, Szabo, D. Thurston

↳ Today

grid diagrams

$K \subset S^3$ knot

Def A grid diagram for K is an $n \times n$ grid in the plane with some marking $X_1, X_2 \cdots X_n$
$O_1, O_2 \cdots O_n$



s.t. in each column & row
$\exists$ exactly one X & one O

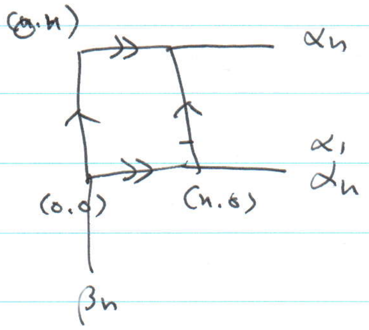
trefoil

If we join X to O in each column & each row
(vertical top)

then we get a diagram for the knot K

Remark All this can be done for links too

G = grid diagram $\rightsquigarrow$ put on a torus



$\alpha_i = \{y = i\} \quad 1 \leqq i \leqq n$

$\beta_j = \{x = j\} \quad 1 \leqq j \leqq n$

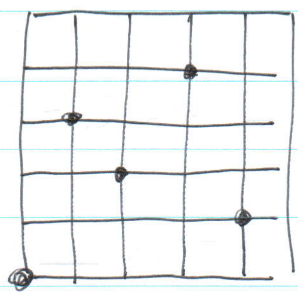
$x_{ij} = \alpha_i \cap \beta_j$

Set of generators

$S(G) = \{\vec{x} = (x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}),\ \sigma \in S_n\} \cong S_n$

We will work over $F = \mathbb{Z}/2$

<u>Rmk</u> everything can be done on $\mathbb{Z}$



$R = \mathbb{F}[U_1, \cdots, U_n]$

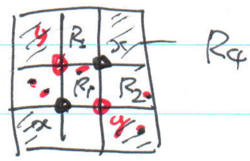
We'll define a complex $(C(G), \partial)$

$C(G)$: free R-module with generators $S(G)$

<u>Notation</u>

If $\vec{x}, \vec{y}$ differ by a transition

i.e. differ on 2 rows & 2 columns only



$\mathcal{T}$ = torus

$= R_1 \cup R_2 \cup R_3 \cup R_4$

$\mathrm{Rect}(\vec{x}, \vec{y}) := \{R_1, R_2\}$ 　 $x \to y$ のときに左側に見る

$\mathrm{Rect}(\vec{y}, \vec{x}) = \{R_3, R_4\}$

$\mathrm{Rect}(\vec{x}, \vec{y}) \ni r$ set $X_i(r) = \begin{cases} 1 & \text{if } x_i \in r \\ 0 & \text{if not} \end{cases}$

$O_i(r) = \begin{cases} 1 & \text{if } O_i \in r \\ 0 & \text{if not} \end{cases}$

$G(r) = \# \{ x_{ij} \in \vec{X} \mid x_{ij} \in \mathrm{Int}(r) \}$

e.g. $G(R_1) = 2$, $G(R_3) = 0$

If $\vec{x}, \vec{y}$ not differed by a transposition

set $\mathrm{Rect}(\vec{x}, \vec{y}) = \phi$

$\mathrm{Rect}^{0}(\vec{x}, \vec{y}) := \{ r \in \mathrm{Rect}(\vec{x}, \vec{y}) \mid G(r) = 0 \}$

$\partial^- : C^-(G) \longrightarrow C^-(G)$ linear

$\vec{x} \in S(G)$ set $\partial^- \vec{x} = \sum_{\vec{y}} \sum_{\substack{r \in \mathrm{Rect}^0(\vec{x}, \vec{y}) \\ X_i(r) = 0 \ \forall i}} u_1^{O_1(r)} u_2^{O_2(r)} \cdots u_n^{O_n(r)} \vec{y}$

count rectangles with no extra generators inside

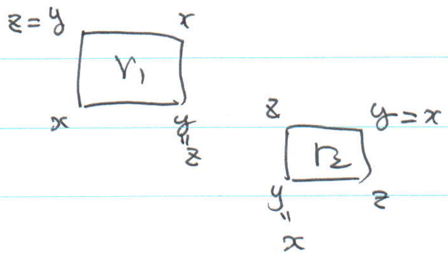
no X's

keep track of O's

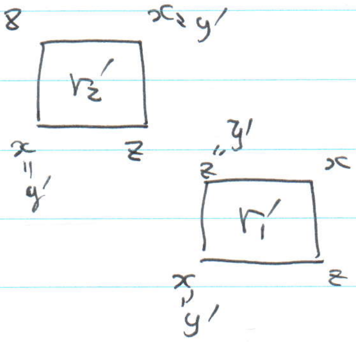
<u>Prop.</u> $\partial^- \circ \partial^- = 0$

(proof) $\partial^- \circ \partial^- x = \sum_{\vec{y}, \vec{z}} \sum_{\substack{r_1 \in \mathrm{Rect}(\vec{x}, \vec{y}) \\ r_2 \in \mathrm{Rect}(\vec{y}, \vec{z}) \\ X_i(r_j) = 0}} u_1^* \cdots u_n^* z$

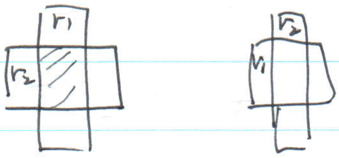
If r_1 & r_2 disjoint

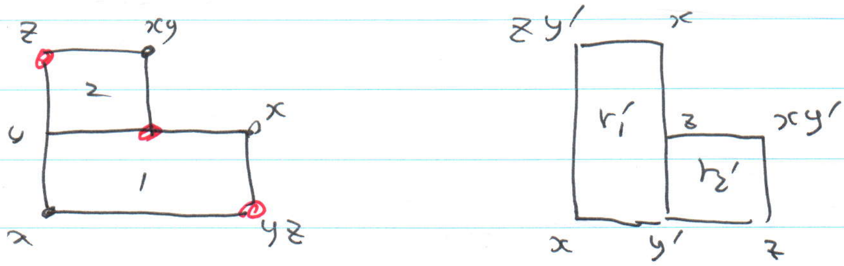


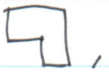
cansels out with



If not





 etc

So $\partial \circ \partial = 0$ //

$C^-(G)$: module over $\mathbb{F}[U_1, \cdots, U_n]$

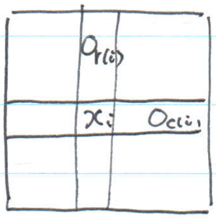
by def.

Lemma. $U_i, U_j : C^-(G) \longrightarrow C^-(G)$ are chain maps

and are chain homotopic

☺ $U_i - U_j = \partial H_{ij} + H_{ij}\partial$

for some $H_{ij} : C^-(G) \circlearrowleft$



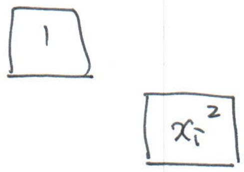
$r, c \in S_n$

It suffice to show

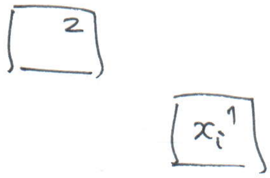
$U_{c(i)}$ & $U_{r(i)}$ are chain homotopic

$$H(\vec{x}) = \sum_{\vec{y}} \sum_{\substack{r \in \mathrm{Rect}(\vec{x}, \vec{y}) \\ X_i(r) = 1 \\ X_j(r) = 0 \ \forall j \neq i}} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \vec{y}$$

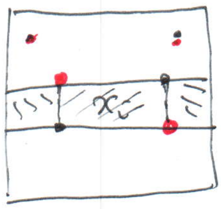
Terms $\partial H + H\partial$ most cansel out



cansels out

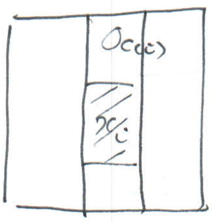


except



contributes $U_{r(i)}\vec{x}$

to $\partial \circ H + H \circ \partial$



$\rightsquigarrow U_{c(i)}\vec{x}$

$$\therefore (\partial H + H\partial)\vec{x} = (U_{r(i)} - U_{c(i)})\vec{x}$$

Def. $H^-(G) = H_*(C^-(G), \partial^-)$

module over $\mathbb{F}[U_1, \dots, U_n]$

but all U_i's act the same way

$=$ module over $\mathbb{F}[U]$ $\quad U = U_i \ \forall i$

Thm $HFK^-(K) = H^-(G)$ is a knot invariant, as a $\mathbb{F}[U]$-module.

proof: later

$\hat{C}(G) = C^-(G) \otimes_R R/U_1$ i.e. set $U_1 = 0$

$(\hat{C}(G), \hat{\partial})$: gives homology group $\widehat{HFK}(K)$

$\mathbb{F}[U_2, \cdots, U_n]$-module

but U_i's act trivially.

$C^-, \hat{C}$ have bigradings (M, A)

Maslov m Alexander s

$A, M : S(G) \to \mathbb{Z}$

<u>relative gradings</u> $A(\vec{x}) - A(\vec{y})$

If $\vec{x}, \vec{y}$: differ by a transposition, $r \in \mathrm{Rect}(\vec{x}, \vec{y})$

$$A(\vec{x}) - A(\vec{y}) = \sum_i X_i(r) - \sum_i O_i(r)$$

indep. of r

$$M(\vec{x}) - M(\vec{y}) = 1 + 2G(r) - 2\sum_i O_i(r)$$

$$A(U_1^{m_1} \cdots U_n^{m_n} \vec{x}) = A(\vec{x}) - (m_1 + \cdots + m_n)$$
$$M(U_1^{m_1} \cdots U_n^{m_n} \vec{x}) = M(\vec{x}) - 2(m_1 + \cdots + m_n)$$

<u>Rem.</u> ∂^- fixes A, drops M by 1

$\boxed{r}$ $\vec{x}$ to $U_1^{m_1} \cdots U_n^{m_n} \vec{y} = \hat{\vec{y}}$

$$A(\vec{x}) = A(\hat{\vec{y}})$$
$$M(\vec{x}) = M(\hat{\vec{y}}) + 1$$

link case $\mathbb{F}[U_1, \cdots, U_\ell]$ ℓ: # of components